METHOD OF DETERMINING PROCESS DYNAMICS

by

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I. INTRODUCTION

There are two basic problems in process dynamics;

1. the prediction of the response of a known process to a known input signal,
2. the determination of the dynamic characteristics of an unknown process through the measurement of its response to a certain signal.

Both these basic problems can be approached by using the same mathematical tools, the mathematics of linear dynamic systems. It is, however, necessary to make some rather drastic assumptions about the dynamic nature of the process under question to use these tools rigorously. The basic assumption is that of local linearity. Most processes are non-linear in nature, hence the applicability of linear analysis is limited to the study of local behavior, under a small disturbance.

Throughout this paper we shall consider a general process consisting of a single input variable and a single output variable related through a linear dynamical process as illustrated in Figure 1.

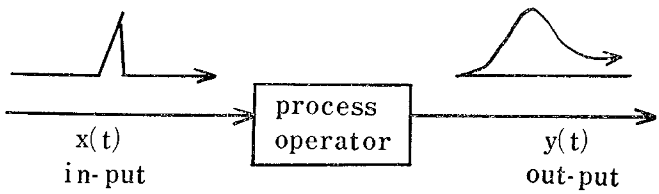

Fig. 1. Block diagram of a linear systems

This process operator takes the mathematical form of an nth-order ordinary linear differential equation with constant coefficients which is denoted by,

$$a_n \frac{dn}{dt^n} y(t)+a_{n-1} \frac{d^{n-1}}{dt^{n-1}} y(t)+\cdots+a_1 \frac{d}{dt} y(t) +a_0 y(t)=x(t) \tag{1}$$

This representation is not always the most convenient to use in practice and three equivalent methods of description can be used, the transfer function, the frequency response and the weighting function method. These alternate methods are discussed in the following sections.

II. The Transfer Function

A considerable body of knowledge has been developed, using the transfer function as a means of describing the dynamics of a process. The transfer functions is used most extensively in the analysis and synthesis of control system. The transfer function is related to the differential equation mathematically by the Laplace transfer operator as shown in the following way.

We multiply both sides of equation (1) by e^{-st} and integrate from $t=0$ to $t=\infty$. Then let denote,

$$\int_0^\infty e^{-st}\ y(t)\ dt = Y(s),$$

$$\int_0^\infty e^{-st}\ x(t)\ dt = X(s)$$

and by partial integration for each term,

$$\int_0^\infty e^{-st}\frac{dy}{dt}dt = -y_0 + s\int_0^\infty e^{-st}\ y(t)\ dt = -y_0 + sY(s)$$

$$\int_0^\infty e^{-st}\frac{d^2y}{dt^2}dt = -y_0^{(1)} - sy_0 + s^2Y(s)$$

$$\vdots$$

$$\int_0^\infty e^{-st}\frac{d^ny}{dt^n}dt = -y_0^{(n-1)} - sy_0^{(n-2)} - \cdots - s^{n-1}y_0 + s^nY(s)$$

where the initial conditions are specified as,

$$y_0 = (y)_{t=0}$$

$$y_0^{(1)} = \left(\frac{dy}{dt}\right)_{t=0}$$

$$\vdots$$

$$y_0^{(n-1)} = \left(\frac{d^{n-1}y}{dt^{n-1}}\right)_{t=0}$$

Then equation (1) together with the initial conditions (2) can be written as,

$$\begin{aligned}(a_ns^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0)Y(s) &= X(s)\\ &+ a_ny_0s^{n-1} + (a_ny_0^{(1)} + a_{n-1}y_0)s^{n-2}\\ &+ (a_ny_0^{(2)} + a_{n-1}y_0^{(1)} + a_{n-2}y_0)s^{n-3} + \cdots\\ &+ (a_ny_0^{(n-1)} + a_{n-1}y_0^{(n-2)} + \cdots + a_1y_0)\end{aligned} \tag{3}$$

Hence if we define the polynominals $D(s)$ and $N(s)$ as,

$$D(s) = a_ns^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 \tag{4}$$

and

$$\begin{aligned}N(s) &= a_ny_0s^{n-1} + (a_ny_0^{(1)} + a_{n-1}y_0)s^{n-2} + \cdots\\ &+ (a_ny_0^{(n-1)} + a_{n-1}y_0^{(n-2)} + \cdots + a_1y_0)\end{aligned} \tag{5}$$

then the Laplace transformed solution of equation (3) can be written in simpler form as,

$$Y(s) = \frac{X(s)}{D(s)} + \frac{N(s)}{D(s)} \tag{6}$$

We note that the second term of the equation depends on the initial condition of which represent the heredity of motion as it stated in the clasic mechanics while the first term represents its new environment under the forcing function $x(t)$ or its transformed function $X(s)$. $N(s)$ is at most of order $(n-1)$th and is thus a lower order than $D(s)$. It will vanish if all the initial values specified by equation (2) vanish. That is the case of asumming steady state exist before the cause of change $x(t)$ is applied. Therfore the output $Y(s)$, the Laplace transformed $y(t)$, is only related to input $X(s)$. The ratio of output to the input is called transfer function $F(s)$, which is characterized by the polynominal $D(s)$ given by equation (4).

Usually, it has specific mathematical form finding from process dynamic analysis, but it does express the signal size and time effect by comparing the incoming and outgoing signal actually, and measure out some characteristic value to fit the standard form of the solution of the differential equation.

The advantage of using transfer function by the Laplace transform method is thus to reduce a problem in differential equation to one of algebraic operation. The step of going from $Y(s)$ to $y(t)$ is seldom necessary, because the behavior of $y(t)$ is fully determined by $Y(s)$. Thus it is possible to translate the engineering requirements on $y(t)$ to a set of requirements on $Y(s)$, or with the input characteristics specified, to a set of requirements on $F(s)$, the transfer function.

The parameter s in the Laplace transform is a complex variable, it can be any value in the complex plane. The most convenient s is the origin. Then,

$$F(0)=K$$

has a physical meaning, that is the size ratio of output to input steady signal, actualy it is the gain of the system.

The transfer function $F(s)$ is essentially determined for any s. Therefor another significant s is on the imaginary axis, $s=j\omega$, where ω is real. For physical systems, the coefficient in the $F(s)$ are all real. Then assuming the complex function of $F(j\omega)$ can be expressed as the sum of its real and imaginary parts.

$$F(j\omega)=u+jv$$

where u and v are function of ω. since real parts will contain ω only as $(j\omega)^2$ or $(j\omega)^4$ and so on, the same u and v can be used to express $F(-j\omega)$ as,

$$F(j\omega)=u-jv$$

because only v which contain $(j\omega)$ or $(j\omega)^3$ and so on will change the sign. Therefore $F(-j\omega)$ is the complex conjugate pair of $F(j\omega)$, so the knowledge of $F(j\omega)$ for $\omega\geq 0$ will be enough to discribe the transfer function. $F(j\omega)$ for all positive value of ω is called the frequency response of the system.

III. Frequency Response Method

The use of frequency response method to describe actual system has gained wide spread popularity because of the simplicity in treating complex system by actual dynamic-response measurements. The complex function $F(j\omega)$ may take rectangular form as $u+jv$ as stated before, or it may write as exponential form, $Me^{j\phi}$ or even in polar form $M\angle\phi$. Where M is the magnitude ratio of output to input, ϕ is the argument of the vector angle for the specific frequency ω. Usually this angle is negative for actual system and may be called the phase lag and omit the minus sign.

The foregoing section shows that frequency response can be easily predicted from transfer function mathematically. But our subject is the reverse—the determination of a complicated transfer function by some kind of response data. Of the several way of plotting frequency-response data, the one is widest current use in the process industry is the Bode plot, Phase angle and magnitude ratio are plotted separatelly versus freqency on a semilog paper.

The most direct method for obtaining a frequency response of a hardware system is to measure the steady-state response to a sinusoidal input of suitable frequencies and amplitudes. If it is suspected that the system is only approximately linear, the amplitude of the wave is kept small and care is taken to keep the absolute level of the signal because of the nonlinearity of the process distor the sine output. Unfortunately, small amplitude leads to low precission to obtain by the direct method.

A more sophisticated frequency-response system employs a test unit that generats a forcing voltage of adjustable frequency, amplitude, and average value; accepts the output of the unit being tested and transduced to voltage; shift its phase in accordance with dial settings; and plots the output against the input. The phase angle is the setting on the phase dial that gives the inphase pattern. Amplitude ratio is read directly from the plot. The pattern of the plot also shows various from linear behavior.

IV. Frequency Response from Pulse Testing

A more practical method of obtaining a frequency response is to analyze the transient response of the process output to a shaped pulse input. A single transient contains the entire frequency spectrum and can be obtained in a matter of seconds. For several hundred years it has been known that all transient wave shape may be considered to be made up of a sum of sinusoidal wave shapes of various amplitudes and covering the entire frequency band. The basic procedure for converting transient data from the time to the frequency domain is based on the use of the Fourier integral which, under certain conditions, enables a time function $g(t)$ transformed into a complex frequency function $G(j\omega)$ as,

$$G(j\omega)=\int_0^\infty g(t)\ e^{-j\omega t}\ dt$$

This integral must be evaluated from time zero to infinity for each frequency ω at which $G(j\omega)$ is desired. The integration can be accomplished only it the behavior of $g(t)$ is known for an infinite time after the disturbance is initiated. This means that it is necessary for the system to reach a steady state in some finite time.

From the definitions of the transfer function,

$$F(s)=\frac{Y(s)}{X(s)}=\frac{\int_0^\infty y(t)e^{-st}dt}{\int_0^\infty x(t)e^{-st}dt}$$

Now to find the frequency response of the system, substitute $j\omega$ for s,

$$F(j\omega)=\frac{\int_0^\infty y(t)e^{-j\omega t}dt}{\int_0^\infty x(t)e^{-j\omega t}dt}$$

For $e^{-j\omega t}$, substitute its equivalent $\cos\omega t-j\sin\omega t$, then,

$$F(j\omega)=\frac{\int_0^\infty y(t)\cos\omega t\ dt-j\int_0^\infty y(t)\sin\omega t\ dt}{\int_0^\infty x(t)\cos\omega t\ dt-j\int_0^\infty x(t)\sin\omega t\ dt} \qquad (7)$$

Now for any value of ω chosen we have an expression for $F(j\omega)$, the response to the system would give to sinusoidal forcing of that frequency. But here it is in the form of two complex numbers (the numerator and the denominator). The real parts R and the imaginary parts I of these two complex numbers are each an integral of the product of a particular sine or cosine function of time times input function $x(t)$ and output function $y(t)$, which is known from measurement. After the numerical integrations have been done, we may express $Y(j\omega)$ in polar form as $A_1\angle\theta_1$, where A_1 is the magnitude of $\sqrt{R_1^2+I_1^2}$ and θ_1 is the phase angle $\tan^{-1}(I/R)$. This computation is repeated for the input $X(j\omega)$ as $A_2\angle\theta_2$ and the unknown system frequency response is then expressed as the ratio of two vector quantities,

$$F(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{A_1}{A_2}e^{j(\theta_1-\theta_2)}=Me^{j\phi} \qquad (8)$$

The magnitude ratio M is the ratio of individual amplitudes, and the phase angle ϕ is the difference between the individual angles.

There are several practical numerical integration methods that may be used in the evaluation of the Fourier integral. One of the most commen is Simpson's area rule which is used to obtain the area under the product curve $g(t)$ cos ωt and $g(t)$ sin ωt by a series of parabolic approximations. To insure resonable accuracy with this method, the time history of $g(t)$ must be evaluated at small enough time intervals $\triangle t$ so that each cycle of the product curve can be defined by at least eight measurements. This means that $\omega\triangle t$, where $\triangle t$ is the time interval between measurements, should not exceed 45 degree.

A refinement to this basic method introduced first by Filion (1) and later applied by Schumacher (2) enables the parabolic approximation to be applied directly to the time function $g(t)$ rather than to the product curve $g(t)$ cos ωt and $g(t)$ sin ωt. This allows greater accuracy in the determination of high-frequency components and it is generally posible to obtain resonable accuracy with $\omega\triangle t$ as high as 120 degree.

However, if the computation process is systematically programmed for execution by a digital computer, all that need be done for a given experimental run is to reduce the input and output curves to two lists of numbers with corresponding time values and feed these into the computer along with a list of ω values to be used. Out will come a list of $|F(j\omega)|$ and $\angle F(j\omega)$ versus ω. There are special equipment called "Transfer Function Analyzer (3)" designed to report the input and out put function in digital form directly.

Regardless of integration method used to obtain a frequency response, the geometric shape of input function has a profound bearing on the frequency range through which reliable transforms can be obtained and it should be given careful consideration. The Fourier transform of the input is an indication of the excitation that is applied to the system at any frequency. Two extreme type of input are the pure step and the pure impulse. The unit step has a transform magnitude equal to $1/\omega$, thus giving infinite excitation to the zero frequency component at the expense of higher frequencies. An impulse, on the other hand, has a constant transform magnitude over the

(1) "On A Quadrature Formula for Trigonometric Integrals" by L. N. G. Filion, Proceedings of the Royal Society of Edinburgh, Scotland, vol. 49 (XLIX) 1928-1929, pp. 38-47.

(2) "Methods of Analyzing Transient Flight Data to Obtain Air-craft Frequency Response" by L. E. S. Schumacher WADC Memo. Rep. No. MCRFT- 2268, January 1950.

(3) The Solartron Electric Group LTD. England, Type JM 1600

entire frequency spectrum. Thus it appears that for most purpose the impulse is a more desirable type of input and that the step input is effective only at very low frequencies. This fact will used in a different method of describing process as it will be discussed in the next section.

V. Weighting Function Method

One last method of describing the dynamics behaviour of a linear process is presented. This method describes the process in terms of its response to a fictional input called the unit impulse. The unit impulse is the limit of a unit pulse of duration ϵ and intensity $1/\epsilon$.

Define $f(\tau)$ as the response of the process to such an impulse at time $\tau=0$. Physically, such a response can be realized only as a limit of the response to a finite unit pulse. But such a function $f(\tau)$ does exist in mathematical sense and can be used to describe the process, which is called weighting function. It shows how much weight that a pulse input can bring the output change.

Consider the value of the output to the process at time t_1 to be caused by a pulse of intensity propotional to $x(\lambda_i)$ the runing time variable of input, duration $\triangle\lambda$. This contribution to output $y(t_1)$ from the input pulse at time τ is $[f(t_1-\lambda_i)]\ [x(\lambda_i)\triangle\lambda]$. The first term is the response to a unit impulse and the second term is the scale factor which corrects for the fact that the impulse is not a unit impulse but of intensity $x(\tau_i)\triangle\lambda$. This summed over all past time give the total value of $y(t_1)$.

$$y(t_1)=\sum_{i=0}^{n} f(t_1-\lambda_i)x(\lambda_i)\triangle\lambda$$

which become on the limit as $\triangle\lambda\to 0$, and this expression is true for all the time t after the disturbance. So, we replace t_1 by t, then,

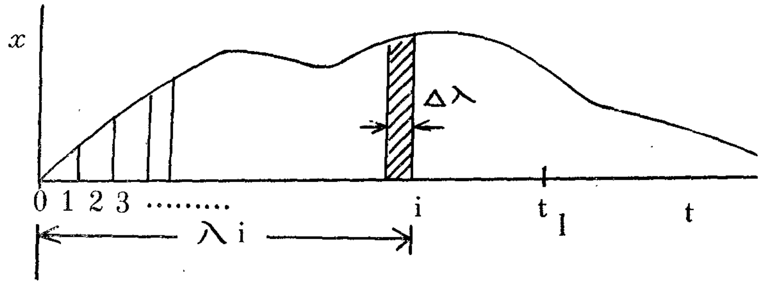

Figure 2.

$$y(t)=\int_0^t f(t-\lambda)x(\lambda)d\lambda \tag{9}$$

This type of an integral is a convolution integral. Equation (9) is a third way of describing the process dynamics.

An alternative form to equation (9) is,

$$y(t)=\int_0^t f(\lambda)x(t-\lambda)d\lambda \tag{10}$$

Since for any physically realizable process $f(\tau)=0$ for $\tau<0$, so forms,

$$y(t)=\int_0^\infty f(\lambda)\ x(t-\lambda)d\lambda=\int_{-\infty}^\infty f(\lambda)x(t-\lambda)d\lambda \tag{11}$$

are also possible expression for this particular convolution integral.

We have seen the convolution integral for discribing the cause and effect relations as equation (11). From statistical approach we have similar expression as

$$\phi_{yx}(\tau)=\int_{-\infty}^{\infty} f(\lambda)\phi_{xx}(\tau-\lambda)d\lambda \qquad (12)$$

Where $\phi_{yx}(\tau)$ represent cross-correlation function of input signal and output response, and $\phi_{xx}(\tau)$ is the self-correlation function of input signal. This equation can be writen by power spectral density,

$$\Phi_{yx}(\omega)=F(j\omega)\cdot\Phi_{xx}(\omega) \qquad (13)$$

where, $\Phi_{yx}(\omega)=\int_{-\infty}^{\infty}\phi_{yx}(\tau)e^{-j\omega}d\tau$

and, $\Phi_{xx}(\omega)=\int_{-\infty}^{\infty}\phi_{xx}(\tau)e^{-j\omega}d\tau$

That is the Fourier transforms of the correlation functions. If we have white noise for input, that is $\Phi_{xx}(\omega)=\Phi_{xx}(0)$=constant. The correlation function is simply as by Lee (4),

$$\phi_{yx}(\tau)=\Phi_{xx}(0)\cdot f(\tau) \qquad (14)$$

But it is impossible to have white noise for the input. Therefore Wiener (5) suggest to use on-off signal with random frequency as test signal, but it still has difficulty to make such a signal actualy, so Huffman (6) make a proposal to use M-sequential time series for this purpose.

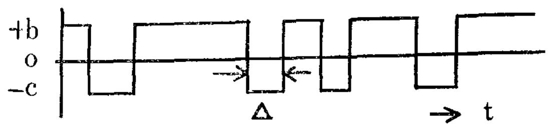

Figure3. M-series

Let consider a period. of 15△ M-series as shown in Figure 3. The noise level of −c to +b is madie from some switching device, and record the output by an integrating recorder with switching relay to comput the correlation function of,

$$\phi_{yx}(\tau)=\frac{1}{N\triangle}\int_{0}^{N\triangle} y(t)x(t-\tau)dt \qquad (15)$$

On the other side the auto-correlation function may be reduced as,

$$\phi_{xx}(\tau)=\frac{1}{N\triangle}\int_{0}^{N\triangle} x(t)x(t-\tau)dt$$

$$\phi_{xx}(\tau)=\frac{1}{N}\left(\frac{N+1}{2}b^2+\frac{N-1}{2}c^2\right)$$

for $\tau=nN\triangle$, and between the period

$$\phi_{xx}(\tau)=\frac{1}{N}\left(\frac{N+1}{4}b^2+\frac{N-3}{4}c^2-\frac{N+1}{2}bc\right)$$

This correlation could be zero if following relation exist,

(4) Y. W. Lee, "Application of Statistical Methods to Communications Problem", MIT Research Laboratory for Electronics Tech. Rep., 181, p.28, 1950.

(5) N. Wiener, "The Spectrum of An Array", Journal of Mathematics and Physics, MIT, vol. 6, p. 151, 1927.

(6) D. A. Huffman, "The Synthesis of Linear Sequential Coding Networks", Information Theory Academic Press. p. 77-95, 1956.

$$\frac{b}{c}=1\pm\frac{2}{\sqrt{N+1}}$$

The Fourier transform of the correlation function will be,

$$\Phi_{xx}(\omega)=\frac{N+1}{N}\left(\frac{b+c}{2}\right)^2\left(\frac{\sin\omega\triangle/2}{\omega\triangle/2}\right)^2\triangle$$

$$\Phi_{xx}(0)=\frac{N+1}{N}\left(\frac{b+c}{2}\right)^2\triangle \qquad (16)$$

Then from equation (14), (15) and (16),

$$f(\tau)=\frac{1}{N\triangle}\int_0^{N\triangle} y(t)x(t-\tau)dt\Big/\frac{N+1}{N}\left(\frac{b+c}{2}\right)^2\triangle$$

According to Furuta and Izawa (7), the test result were in fairly good coincidence with the theory, and this is very useful and promises future applications in adaptive control systems.

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(7) Furuta and Izawa, "Method of Determining Process Dynamics", Journal of The SICE, Japan, vol. 3, p. 665, 1964.

程序動態之決定法

郭 文 東

摘要：著者站在分析工廠實際程序之立場，比較其四種動態特性之表示法，作如下之結論。

1. 直接應用微分方程，來描寫程序之動態，必須具備有該過程之全盤智識；而且解方其程亦不易。此法只提供基本觀念而已。
2. 最好將方程演變成因果間之傳遞函數。但爲得其函數仍然要分析該過程之內容。
3. 頻率反應法只要分析進出之波形，不必追問過程本身，就得以推測該程序之特性。在電學具有廣範之用途。但在一般的工廠程序，則反應遲鈍；又比例缺佳，很難期待其普遍應用。
4. 對進出該程序之不規則信號，施以相關分析，最有將來性。但其計算繁雜，非人工所能勝任，須要借助於電子計算機。

本文强調用M信號作試驗，以類比計算機作相關分析之方法，簡單而可實行。

METHOD OF DETERMINING PROCESS DYNAMICS

Wen-tung Kuo

Summary

This paper presents a comparison of four methods for expressing process dynamics applied to the analysis of a production process.

1. The approach using differential equations needs complete knowledge about the whole process under discussion, but it will give only a basic idea unless it has a solution.
2. Laplace transformed transfer function expresses a process most clearly, but it still requires the theoretical analysis of the process.
3. The frequency response method does not require the theoretical analysis of the process, so it is widely used for testing the electrical component or system. However, it has only a limited use in process analysis because of the slow response and narrow proportionality in the nature of the process.
4. Correlation function analysis has a promising future in this field, but it requires data-processing equipment.

The author has put more emphasis on the last method by introducing M-sequential time series testing.